

# The conditions for quantum violation of macroscopic realism

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Why do we not experience a violation of macroscopic realism in every-day life? Normally, no violation can be seen either because of decoherence or the restriction of coarse-grained measurements, transforming the time evolution of any quantum state into a classical time evolution of a statistical mixture. We find the sufficient condition for these classical evolutions for spin systems under coarse-grained measurements. Then we demonstrate that there exist "non-classical" Hamiltonians whose time evolution cannot be understood classically, although at every instant of time the quantum spin state appears as a classical mixture. We suggest that such Hamiltonians are unlikely to be realized in nature because of their high computational complexity.

The laws of quantum physics are in conflict with a classical world, in particular with local and macroscopic realism as characterized by the violation of the Bell [1] and Leggett-Garg [2, 3] inequality, respectively. While Bell's theorem is a well investigated area of research, hardly any analysis has been undertaken to understand the key ingredients for the violation of macroscopic realism (macrorealism). Is it the initial state, the Hamiltonian or the measurement observables which have to be "quantum" to see a deviation from classical physics?

Macrorealism is defined by the conjunction of three postulates [3]: "(1) *Macrorealism per se*. A macroscopic object which has available to it two or more macroscopically distinct states is at any given time in a definite one of those states. (2) *Non-invasive measurability*. It is possible in principle to determine which of these states the system is in without any effect on the state itself or on the subsequent system dynamics. (3) *Induction*. The properties of ensembles are determined exclusively by initial conditions (and in particular not by final conditions)." These assumptions allow to derive Leggett-Garg inequalities.

In this Letter we first show that a violation of the Leggett-Garg inequality itself is possible for *arbitrary* Hamiltonians given the ability to distinguish consecutive eigenstates. This is understandable because it is generally accepted that "microscopically distinct states" do not have objective existence. For testing macrorealism one needs to apply the Leggett-Garg definition referring to macroscopically distinct states. In our every-day life, to experience macrorealism it is usually sufficient to employ a certain type of decoherence (where the system is *isolated* [16] and only at the times of measurement the environment makes a pre-measurement on the apparatus [4]) or the restriction of coarse-grained measurements [5, 6, 7, 8]. While both mechanisms transform the quantum state at every instance of time into a classical mixture, we demonstrate that there are "non-classical" Hamiltonians for which the *time evolution of this mixture cannot be understood classically*, leading to a violation of macrorealism. We find the necessary condition for non-classical evolutions and illustrate it by the example of a Schrödinger cat-like state [9]. In the last part we argue why such Hamiltonians are unlikely to be realized.

Consider a physical system and a quantity  $A$ , which whenever measured is found to take one of the values  $\pm 1$  only. Now perform a series of runs starting from identical initial conditions (at time  $t = 0$ ) such that on the first set of runs  $A$  is measured only at times  $t_1$  and  $t_2$ , only at  $t_2$  and  $t_3$  on the second, and at  $t_1$  and  $t_3$  on the third ( $0 \leq t_1 < t_2 < t_3$ ). Introducing temporal correlation functions  $C_{ij} \equiv \langle A(t_i) A(t_j) \rangle$ , any macrorealistic theory predicts Leggett-Garg inequalities, for instance of the Wigner type [10]:

$$K \equiv C_{12} + C_{23} - C_{13} \leq 1. \quad (1)$$

Any *non-trivial (time-independent) Hamiltonian*  $\hat{H}$  leads to a violation of this inequality. We extend the approach of Peres in Ref. [5] and look at the "survival probability" of the system's initial state at time  $t = 0$ . This state be denoted as  $|\psi(0)\rangle \equiv |\psi_0\rangle$  (which must not be an energy eigenstate) and, without measurements, it evolves to  $|\psi(t)\rangle = \exp(-i\hat{H}t/\hbar)|\psi_0\rangle$  according to the Schrödinger equation. Our dichotomic observable is  $\hat{A} \equiv 2|\psi_0\rangle\langle\psi_0| - \mathbb{1}$ , i.e. we ask whether the system is (still) in the state  $|\psi_0\rangle$  (outcome '+'  $\equiv +1$ ) or not (outcome '-'  $\equiv -1$ ). The temporal correlations  $C_{ij}$  can be written as  $C_{ij} = p_{i+}q_{j+|i+} + p_{i-}q_{j-|i-} - p_{i+}q_{j-|i+} - p_{i-}q_{j+|i-}$ , where  $p_{i+}$  ( $p_{i-}$ ) is the probability for measuring '+' ('-') at  $t_i$  and  $q_{j|ik}$  is the probability for measuring  $l$  at  $t_j$  given that  $k$  was measured at  $t_i$  ( $k, l = +, -$ ). For simplicity we choose  $t_1 = 0$  and equidistant times  $\Delta t \equiv t_2 - t_1 = t_3 - t_2$ . Then the correlation  $C_{12}$  is given by  $C_{12} = 2p(\Delta t) - 1$ , where  $p(t) \equiv |\langle\psi_0|\psi(t)\rangle|^2$  is the (survival) probability to find  $|\psi_0\rangle$  given the state  $|\psi(t)\rangle$ . Analogously, we find  $C_{13} = 2p(2\Delta t) - 1$  and  $C_{23}$ . Plugging everything into (1), one ends up with

$$K = 4p(\Delta t)\sqrt{p(2\Delta t)}\cos\gamma - 4p(2\Delta t) + 1 \leq 1, \quad (2)$$

where  $\gamma \equiv 2\alpha - \beta$  and  $\alpha$  and  $\beta$  are the phases in  $\langle\psi_0|\psi(t_2)\rangle = \sqrt{p(\Delta t)}e^{i\alpha}$  and  $\langle\psi_0|\psi(t_3)\rangle = \sqrt{p(2\Delta t)}e^{i\beta}$ .

Now, independent of the system's dimension, it is sufficient to consider as initial state a superposition of only two energy eigenstates  $|u_1\rangle$  and  $|u_2\rangle$  with energy eigenvalues  $E_1$  and  $E_2$ , respectively:  $|\psi_0\rangle = (|u_1\rangle + |u_2\rangle)/\sqrt{2}$ . Ineq. (2) becomes  $K = 2\cos(\frac{\Delta E\Delta t}{\hbar}) - \cos(\frac{2\Delta E\Delta t}{\hbar}) \leq 1$ , with  $\Delta E \equiv E_2 - E_1$  the energy difference of the two levels, and a violation is always possible.

The left hand side reaches  $K = 1.5$  for  $\Delta t = \frac{\pi\hbar}{3\Delta E}$  and  $\Delta t = \frac{5\pi\hbar}{3\Delta E}$  and in  $\frac{2\pi\hbar}{\Delta E}$  periods thereof.

Why then do we not see a violation of the Leggett-Garg inequality in everyday life? The usual answer is that this is either due to decoherence or due to the fact that the resolution of our everyday measurements is not sharp, making it impossible to project onto individual states and hence making it impossible to see the above demonstrated violation that is always present for microstates.

For testing macrorealism—i.e. testing the Leggett-Garg inequality under the restriction of *coarse-grained measurements*—we consider a spin- $j$  system (with  $j \gg 1$ ) as a model example. Any spin- $j$  state can be written in the quasi-diagonal form  $\hat{\rho} = \iint P(\Omega) |\Omega\rangle\langle\Omega| d^2\Omega$  with  $d^2\Omega$  the solid angle element and  $P$  a normalized and *not necessarily positive* real function [11]. The spin coherent states  $|\Omega\rangle \equiv |\vartheta, \varphi\rangle$ , with  $\vartheta$  and  $\varphi$  the polar and azimuthal angle, are the eigenstates with maximal eigenvalue of a spin operator pointing into the direction  $\Omega \equiv (\vartheta, \varphi)$  [12]:  $\hat{\mathbf{J}}_\Omega |\Omega\rangle = j|\Omega\rangle$  in units where  $\hbar = 1$ . In coarse-grained measurements our resolution is not able to resolve individual eigenvalues  $m$  of a spin component, say the  $z$ -component  $\hat{J}_z$ , but bunches together  $\Delta m$  neighboring [17] outcomes into “slots”  $\bar{m}$ , where the measurement coarseness is much larger than the intrinsic uncertainty of coherent states, i.e.  $\Delta m \gg \sqrt{j}$  [8].

The question arises whether it is problematic to use coarse-grained *von Neumann measurements* of the form  $\sum_{m \in \{\bar{m}\}} |m\rangle\langle m|$ , where  $|m\rangle$  are the  $\hat{J}_z$  eigenstates, as “classical measurements”. In contrast to the *positive operator value measure* (POVM), they have sharp edges and could violate the Leggett-Garg inequality by distinguishing with certainty between microstates at two sides of a slot border. Therefore, we model our coarse-grained  $\hat{J}_z$  measurements as belonging to a (spin coherent state) POVM, where the element corresponding to the outcome  $\bar{m}$  is represented by

$$\hat{P}_{\bar{m}} \equiv \frac{2j+1}{4\pi} \iint_{\Omega_{\bar{m}}} |\Omega\rangle\langle\Omega| d^2\Omega. \quad (3)$$

Here,  $\Omega_{\bar{m}}$  is the angular region of polar angular size  $\Delta\Theta_{\bar{m}} \sim \Delta m/j \gg 1/\sqrt{j}$  whose projection onto the  $z$  axis corresponds to the slot  $\bar{m}$ . As the  $\Omega_{\bar{m}}$  are mutually disjoint and form a partition of the whole angular region, we have  $\sum_{\bar{m}} \hat{P}_{\bar{m}} = \mathbb{1}$ . The POVM elements are overlapping at the slot borders over the angular size  $\sim 1/\sqrt{j}$  which is small compared to the angular slot size  $\Delta\Theta_{\bar{m}}$ . In the basis of  $\hat{J}_z$  eigenstates  $\hat{P}_{\bar{m}} = \sum_{k=-j}^j \frac{2j+1}{4\pi} \iint_{\Omega_{\bar{m}}} |k\rangle\langle k| d^2\Omega$  is diagonal where  $|k\rangle\langle k| = \binom{2j}{j+k} \cos^{2(j+k)} \frac{\vartheta}{2} \sin^{2(j-k)} \frac{\vartheta}{2}$ .

The probability for getting the particular outcome  $\bar{m}$  is given by  $w_{\bar{m}} = \text{Tr}[\hat{\rho} \hat{P}_{\bar{m}}] = \frac{2j+1}{4\pi} \iint \langle\Omega| \hat{\rho} \hat{P}_{\bar{m}} |\Omega\rangle d^2\Omega$ . This probability can (exactly) be computed via integration of an *ensemble of classical spins* over the region  $\Omega_{\bar{m}}$ , i.e.  $w_{\bar{m}} = \iint_{\Omega_{\bar{m}}} Q(\Omega) d^2\Omega$ , with a *positive probability distribution* (the well-know  $Q$ -function [13]):

$$Q(\Omega) \equiv \frac{2j+1}{4\pi} \langle\Omega| \hat{\rho} |\Omega\rangle. \quad (4)$$

That shows that under fuzzy measurements any quantum state

allows a classical description (i.e. a hidden variable model). This is *macrorealism per se*.

Upon a coarse-grained measurement with outcome  $\bar{m}$ , the state  $\hat{\rho}$  is reduced to  $\hat{\rho}_{\bar{m}} = \hat{M}_{\bar{m}} \hat{\rho} \hat{M}_{\bar{m}}^\dagger / w_{\bar{m}}$  where we have chosen a particular (optimal [5]) implementation of the POVM with the Hermitean Kraus operators  $\hat{M}_{\bar{m}} = \hat{M}_{\bar{m}}^\dagger = \sum_{k=-j}^j \left( \frac{2j+1}{4\pi} \iint_{\Omega_{\bar{m}}} |k\rangle\langle k| d^2\Omega \right)^{1/2} |k\rangle\langle k|$  satisfying  $\hat{M}_{\bar{m}}^2 = \hat{P}_{\bar{m}}$ . We note that, independently of the implementation, the  $\hat{P}_{\bar{m}}$  (and the Kraus operators) behave almost as projectors for all states  $|\Omega\rangle$  except for those near a slot border. In a proper classical limit ( $\sqrt{j}/\Delta m \rightarrow 0$ ) the relative weight of these  $\Omega$  compared to the whole sphere surface becomes *vanishingly small*. The  $Q$ -distribution before the measurement is the (weighted) *mixture* of the  $Q$ -distributions  $Q_{\bar{m}}(\Omega) = \frac{2j+1}{4\pi} \langle\Omega| \hat{\rho}_{\bar{m}} |\Omega\rangle$  of the possible reduced states:

$$Q(\Omega) \approx \sum_{\bar{m}} w_{\bar{m}} Q_{\bar{m}}(\Omega). \quad (5)$$

The approximate sign “ $\approx$ ” reflects that, depending on the density matrix  $\hat{\rho} \equiv \sum_n \sum_{n'} c_{nn'} |n\rangle\langle n'|$ , this relationship may only approximately hold for the set of those  $\Omega \equiv (\vartheta, \varphi)$  near a slot border. In detail eq. (5) reads:  $\frac{2j+1}{4\pi} \sum_n \sum_{n'} c_{nn'} \langle\Omega| n\rangle\langle n'| \Omega\rangle \approx \frac{2j+1}{4\pi} \sum_n \sum_{n'} c_{nn'} [\sum_{\bar{m}} \sqrt{g_{\bar{m}}(n) g_{\bar{m}}(n')}] \langle\Omega| n\rangle\langle n'| \Omega\rangle$  with  $g_{\bar{m}}(k) \equiv \frac{2j+1}{4\pi} \iint_{\Omega_{\bar{m}}} |k\rangle\langle k| d^2\Omega$  which is smaller or equal to 1. Deviations only occur if  $n, n'$  ( $n' \neq n$ ) and  $j \cos \vartheta$  are all within a distance of order  $\sqrt{j}$  to each other and to a slot border [18]. Even in the case of a spin coherent state exactly on a slot border, the overlap between the left and right hand side of eq. (5) is  $\approx 0.997$  (independent of  $j$ ), where the overlap of two probability distributions  $f$  and  $g$  is defined as  $\iint \sqrt{f(\Omega) g(\Omega)} d^2\Omega \in [0, 1]$ . Eq. (5) thus shows that a fuzzy measurement can be understood classically as reducing the previous ignorance about predetermined properties of the spin system [8].

Consider the initial distribution of classical spins,  $Q(\Omega, t_0)$ , corresponding to an initial quantum state  $\hat{\rho}(t_0)$ . We first compute the  $Q$ -distribution of the state  $\hat{\rho}(t_j)$  for an undisturbed evolution without measurement until some time  $t_j$ ,  $Q(\Omega, t_j) = \frac{2j+1}{4\pi} \langle\Omega| \hat{\rho}(t_j) |\Omega\rangle$ . This has to be compared with the *mixture* of all possible reduced distributions upon measurement at a time  $t_i$  ( $t_0 \leq t_i < t_j$ ) with outcomes  $\bar{m}$  which evolved to  $t_j$ , denoted as  $Q_{\bar{m}, t_i}(\Omega, t_j) = \frac{2j+1}{4\pi} \langle\Omega| \hat{U}_{t_j-t_i} \hat{M}_{\bar{m}} \hat{\rho}(t_i) \hat{M}_{\bar{m}}^\dagger \hat{U}_{t_j-t_i}^\dagger |\Omega\rangle / w_{\bar{m}, t_i}$  with  $w_{\bar{m}, t_i} \equiv \text{Tr}[\hat{\rho}(t_i) \hat{P}_{\bar{m}}]$  and  $\hat{U}_t \equiv \exp(-i\hat{H}t)$  the time evolution operator. The system evolves macrorealistically if these two quantities coincide for all  $t_i$  and  $t_j$ ,

$$Q(\Omega, t_j) \approx \sum_{\bar{m}} w_{\bar{m}, t_i} Q_{\bar{m}, t_i}(\Omega, t_j). \quad (6)$$

This is *non-invasive measurability* together with *induction*.

In a dichotomic scenario the outcomes ‘+’ and ‘−’ correspond to finding the spin system in one out of only two slots  $\bar{m} = \pm 1$ . This is represented by a measurement of two complementary regions  $\Omega_+$  and  $\Omega_-$  (for instance the northern and southern hemisphere in a “which hemisphere” measurement). Then, e.g., the probability for measuring ‘−’ at  $t_3$  if ‘+’ was measured at  $t_1$  is given by  $q_{3-|1+} = \iint_{\Omega_-} Q_{+, t_1}(\Omega, t_3) d^2\Omega$  with

$Q_{+,t_1}(\Omega, t_3)$  the  $Q$ -distribution of the state which was reduced at  $t_1$  with outcome '+' and evolved to  $t_3$ . If condition (6) is satisfied, it implies that the probabilities can be decomposed into "classical paths". This means that, e.g.,  $q_{3-|1+}$  is just the sum of the two possible paths via '+' and '-' at  $t_2$ :  $q_{3-|1+} = q_{2+|1+} q_{3-|2+,1+} + q_{2-|1+} q_{3-|2-,1+}$ , where  $q_{3-|2\pm,1+}$  denotes the probability to measure '-' at  $t_3$  given that '+' was measured at  $t_1$  and ' $\pm$ ' at  $t_2$ . Thus, eq. (6) allows to derive Leggett-Garg inequalities such as (1).

We can now establish the *sufficient condition for macrorealism* that holds even for isolated systems, namely

$$\hat{P}_{\bar{m}} \hat{U}_t |\Omega\rangle \approx \begin{cases} \hat{U}_t |\Omega\rangle & \text{for one } \bar{m}, \\ \mathbf{0} & \text{for all the others,} \end{cases} \quad (7)$$

for all  $t$  and  $\Omega$ , allowing deviations at slot borders. This means that  $\hat{U}_t$  does not produce superpositions of macroscopically distinct states and therefore  $\hat{P}_{\bar{m}}$ , and hence  $\hat{M}_{\bar{m}}$ , quasi-behave as projectors. Eq. (7) implies  $\langle \Omega | \hat{U}_{t_j-t_i} \hat{\rho}(t_i) \hat{U}_{t_j-t_i}^\dagger |\Omega\rangle \approx \sum_{\bar{m}} \langle \Omega | \hat{U}_{t_j-t_i} \hat{M}_{\bar{m}} \hat{\rho}(t_i) \hat{M}_{\bar{m}} \hat{U}_{t_j-t_i}^\dagger |\Omega\rangle$  which directly leads to eq. (6). Thus, eq. (7)  $\rightarrow$  eq. (6)  $\rightarrow$  macrorealism.

We denote those Hamiltonians for which eq. (7) is satisfied under coarse-grained measurements as *classical*. An example is the rotation, say around  $x$ ,  $\hat{H} = \omega \hat{J}_x$ , with  $\hat{J}_x$  the spin  $x$ -component and  $\omega$  the angular precession frequency, which satisfies eq. (7) and moreover allows a Newtonian description of the time evolution [8]. But there is no *a priori* reason why all Hamiltonians should satisfy eq. (7). Can one find *non-classical* Hamiltonians violating macrorealism despite coarse-grained measurements? The necessary condition for this is that the Hamiltonian builds up coherences between states belonging to different slots. One explicit (extreme) example is

$$\hat{H} = i\omega (|-j\rangle\langle +j| - |+j\rangle\langle -j|), \quad (8)$$

which, given the special initial state  $|\Psi(0)\rangle = |+j\rangle$ , produces a time-dependent Schrödinger cat-like superposition of two *distant* (orthogonal) spin- $j$  coherent states  $|+j\rangle$  and  $|-j\rangle$ :

$$|\Psi(t)\rangle = \cos(\omega t) |+j\rangle + \sin(\omega t) |-j\rangle. \quad (9)$$

Under fuzzy measurements or pre-measurement decoherence [4], the state (9) appears like a statistical mixture at every instance of time:

$$\hat{\rho}_{\text{mix}}(t) = \cos^2(\omega t) |+j\rangle\langle +j| + \sin^2(\omega t) |-j\rangle\langle -j|. \quad (10)$$

While the two states  $\hat{\rho}_{\text{sup}}(t) \equiv |\Psi(t)\rangle\langle\Psi(t)|$  and  $\hat{\rho}_{\text{mix}}(t)$ , having different  $P$ -functions (Fig. 1), can be distinguished by sharp measurements, they are equivalent on the coarse-grained level. The  $Q$ -distributions,  $Q_{\text{sup}}$  for  $\hat{\rho}_{\text{sup}}(t)$  and  $Q_{\text{mix}}$  for  $\hat{\rho}_{\text{mix}}(t)$ , are given by eq. (4). The coherence terms stemming from  $\hat{\rho}_{\text{sup}}(t)$  are of the form  $\langle \Omega | +j\rangle\langle -j | \Omega \rangle$  and vanish exponentially fast with the spin length  $j$  for all  $\Omega$ . For  $j \gg 1$  the  $Q$ -distributions are practically identical, i.e.  $Q_{\text{sup}}(\Omega, t) \approx Q_{\text{mix}}(\Omega, t) = \frac{2j+1}{4\pi} [\cos^2(\omega t) \cos^4 j(\frac{\Theta_1}{2}) + \sin^2(\omega t) \cos^4 j(\frac{\Theta_2}{2})]$ , where  $\Theta_1 = \vartheta$  ( $\Theta_2 = \pi - \vartheta$ ) is the angle between  $\Omega \equiv (\vartheta, \varphi)$

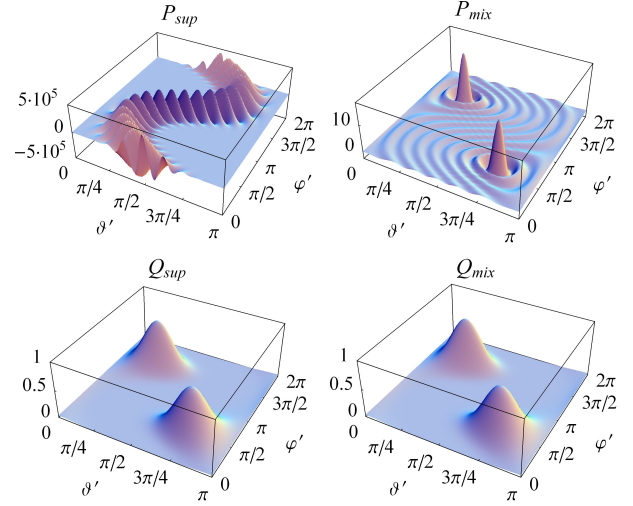


FIG. 1: (Color online.) Top left: The wildly oscillating  $P$ -function  $P_{\text{sup}}$  at time  $t = \frac{\pi}{4\omega}$  of the equal-weight superposition (9) of two opposite spin coherent states  $|+j\rangle$  and  $|-j\rangle$  for spin length  $j = 10$ , plotted in a rotated coordinate system in which  $|+j\rangle = |\frac{\pi}{4}, \frac{3\pi}{2}\rangle$ . Top right: The  $P$ -function  $P_{\text{mix}}$  of the corresponding statistical mixture (10). Bottom: In every-day life the angular measurement resolution is much weaker than  $1/\sqrt{j}$ . Then we cannot distinguish anymore between the superposition state and the classical mixture, as both lead to the same (positive)  $Q$ -distribution  $Q_{\text{sup}} \approx Q_{\text{mix}}$ . Nevertheless, the time evolution of such a mixture can violate macrorealism even under classical (coarse-grained) measurements.

and  $+z$  ( $-z$ ). The  $P$  and  $Q$ -functions of  $\hat{\rho}_{\text{sup}}$  and  $\hat{\rho}_{\text{mix}}$  at  $t = \frac{\pi}{4\omega}$  are shown in Fig. 1 for a certain choice of parameters [19]. Using a dichotomic "which hemisphere" measurement, the temporal correlation function reads  $C_{ij} \approx \cos[\omega(t_j - t_i)]$ . The system effectively behaves as a spin- $\frac{1}{2}$  particle and violates macrorealism. In agreement, eqs. (6) and (7) are not fulfilled. To get macrorealism one would have to coarse-grain always those states which are *connected* by the Hamiltonian and not necessarily in real space. In the present case it is (at least) the outcomes '+ $j$ ' and '- $j$ ' which have to be coarse-grained into one and the same slot, which is of course highly counter-intuitive. Such a coarse-graining would lead to a different kind of macrorealistic physics than the classical laws we know, bringing systems through space and time continuously.

Finally, we suggest a possible reason why non-classical evolutions might be unlikely to be realized by nature: Such evolutions either require Hamiltonians with many-particle interactions or a specific sequence of a large number of computational steps if only few-particle interactions are used ("high computational complexity"). Both cases intuitively seem to be of very low probability to happen spontaneously. Consider our spin- $j$  as a macroscopic ensemble of  $N$  spin- $\frac{1}{2}$  particles (i.e. qubits) such as, e.g., any magnetic material is constituted by many individual microscopic spins. For violating macrorealism it is necessary to build up superpositions of two macroscopically distinct coherent states [20]. Without loss of gen-

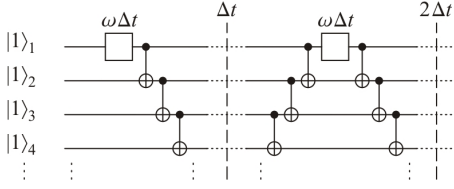


FIG. 2: To simulate the time evolution (11) of a chain of  $N$  qubits one needs  $O(N)$  computational steps per time interval  $\Delta t$ .

erality we consider again the particular Hamiltonian (8). If  $|1\rangle$  and  $|0\rangle$  denote the individual qubit states 'up' and 'down' along  $z$ , then  $|11\dots 1\rangle$  and  $|00\dots 0\rangle$  form the total coherent states  $|+j\rangle$  and  $|-j\rangle$ . The Hamiltonian represents  $N$ -particle interactions of the form  $\hat{H} = \frac{i}{2}(\hat{\sigma}_-^{\otimes N} - \hat{\sigma}_+^{\otimes N})$  where  $\hat{\sigma}_\pm \equiv \hat{\sigma}_x \pm i\hat{\sigma}_y$  with  $\hat{\sigma}_x$  and  $\hat{\sigma}_y$  the Pauli operators. As an alternative one can simulate the evolution governed by this many-body interaction by means of a series of (in nature typically appearing) few-qubit interactions (gates), using the methods of quantum computation science [14]. The task is to simulate

$$|11\dots 1\rangle \rightarrow \cos(\omega t)|11\dots 1\rangle + \sin(\omega t)|00\dots 0\rangle. \quad (11)$$

Assuming sequential qubit interactions, we start from the state  $|11\dots 1\rangle$  and rotate the first qubit '1' by a small angle  $\omega\Delta t$ :  $|1\rangle_1 \rightarrow \cos(\omega\Delta t)|1\rangle_1 + \sin(\omega\Delta t)|0\rangle_1$ . Then we perform a controlled-not (c-not) gate between this qubit '1' and qubit '2' such that  $|x\rangle_1|y\rangle_2 \rightarrow |x\rangle_1|x\oplus y\rangle_2$  ( $x, y = 0, 1$ ). Afterwards c-nots between qubits are performed such that all other qubits are reached (Fig. 2). This procedure brings us to the state at time  $\Delta t$ :  $|11\dots 1\rangle \rightarrow \cos(\omega\Delta t)|11\dots 1\rangle + \sin(\omega\Delta t)|00\dots 0\rangle$ . To simulate the next time interval  $\Delta t$ , we have to undo all the c-nots, rotate the first qubit again by  $\omega\Delta t$ , and make all the c-nots again, leading to the correct state at time  $2\Delta t$ . With this procedure we get a sequence of states, simulating the evolution (11). One needs  $O(N)$  computational steps per interval  $\Delta t$  [21]. Note for comparison, however, that the rotation (say around  $x$ ),  $\hat{H} = \frac{\omega}{2}\sum_{k=1}^N \hat{\sigma}_x^{(k)}$  with  $k$  labeling the qubits, does not require interactions between qubits. Moreover, the simulation of an interval  $\Delta t$  of a spin rotation of the whole chain, i.e.  $|111\dots\rangle \rightarrow [\cos(\omega\Delta t)|1\rangle + \sin(\omega\Delta t)|0\rangle]^{\otimes N}$ , can be achieved in a *single global transformation* on all qubits simultaneously. While both evolutions are rotations in Hilbert space (and require only polynomial resources), the simulation of the "non-classical" cosine-law between states that are distant in real space is—for macroscopically large  $N$ —computationally much more complex than the "classical" rotation in real space [22].

**Conclusion.**—Under sharp measurements any non-trivial Hamiltonian is in conflict with a classical time evolution. Under coarse-grained measurements any quantum spin state appears as a statistical mixture of spins at every instance of time. For classical Hamiltonians these mixtures have a classical time evolution and satisfy macrorealism. Non-classical Hamiltonians build up quantum coherences between macroscopically distinct states, leading to a violation of macrore-

alism. Such Hamiltonians, however, require interactions between a large number of particles or are computationally much more complex than classical Hamiltonians, which might be the reason why they are unlikely to appear in nature.

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  - [16] Here we do not consider decoherence models where the system is continuously monitored by the environment.
  - [17] The term "neighboring" only makes sense in a classical context, treating those eigenvalues as close which correspond to close outcomes in real phase space. In Hilbert space  $|m\rangle$  and  $|m+1\rangle$  are as orthogonal as  $|m\rangle$  and  $|m+10^{10}\rangle$ .
  - [18] If they are not close to each other, the quantity  $\langle\Omega|n\rangle\langle n'|\Omega\rangle$  is exponentially small and suppression by the factor  $\sum_{\bar{m}}\sqrt{g_{\bar{m}}(n)}g_{\bar{m}}(n')$  is not important. If  $n, n'$  are well within a slot  $\sum_{\bar{m}}\sqrt{g_{\bar{m}}(n)}g_{\bar{m}}(n')$  is almost identical to 1.
  - [19] Ref. [13] shows how to compute the  $P$ -functions.
  - [20] For large  $j$  their angular separation  $\Delta\theta$  can be very small and only has to obey the coarse-graining condition  $\Delta\theta \gg 1/\sqrt{j}$ . This guarantees quasi-orthogonality as their (modulus square) overlap is  $\cos^{4j}(\Delta\theta/2) \sim \exp(-j\Delta\theta^2)$ .
  - [21] This is known to be optimal in the case where qubits can interact only sequentially [15]. Relaxing this condition and permitting simultaneous two-qubit interactions, allows to decrease the number of sequential steps but does not change the total number  $O(N)$  of necessary gates per interval.
  - [22] One should, however, mention the possibility that an *external* field may produce an effectively simple non-classical Hamiltonian for  $N$  qubits where the field interacts with the collective modes  $|+j\rangle = |11\dots 1\rangle$  and  $|-j\rangle = |00\dots 0\rangle$ .